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Null Spaces of Differential Operators, Polar Forms, and Splines

DAN GONSOR

Department of Mathematics & Computer Science, Kent State University, Kent, Ohio 44242
E-mail: dgonsor@mcs.kent.edu

AND

MARIAN NEAMTU

Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37240
E-mail: neamtu@orpheus.cas.vanderbilt.edu

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In this article we consider a class of functions, called $\mathscr{D}$-polynomials, which are contained in the null space of certain second order differential operators with constant coefficients. The class of splines generated by these $\mathscr{D}$-polynomials strictly contains the polynomial, trigonometric, and hyperbolic splines. The main objective of this paper is to present a unified theory of this class of splines via the concept of a polar form. By systematically employing polar forms, we extend essentially all of the well-known results concerning polynomial splines. Among other topics, we introduce a Schoenberg operator and define control curves for these splines. We also examine the knot insertion and subdivision algorithms and prove that the subdivision schemes converge quadratically.

1. INTRODUCTION

In recent years *polar forms* have been widely used to describe various properties of *polynomial* splines, including algorithms for their evaluation and numerical manipulation (see e.g., [4, 9, 15, 20, 21]).

In the past a number of non-polynomial splines have been introduced that are known to share many excellent properties with the polynomial splines. In particular, *trigonometric* splines introduced by Schoenberg in 1964 [16] and later investigated by, among others, Lyche and Winther [14], have turned out to have a similar structure. In addition to other desirable properties, certain recurrence relations have been discovered for

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their stable evaluation. Analogous results have also been established for *hyperbolic* splines, introduced by Schumaker in 1983 [19]. In this article we consider a class of splines which contain the polynomial, trigonometric and hyperbolic splines as a special case.

The main objective of the paper is to present a unified theory of this class of splines via the concept of a polar form, introduced in [10]. By systematically employing polar forms, we extend essentially all of the well-known results concerning polynomial splines. While many of the generalizations are straightforward, a number of them seem to be new. The paper generalizes and is in the spirit of the results found in [9, 15, 21] for the polynomial case and [1, 11, 12] for the trigonometric case.

We begin the paper by considering the null space of a second order constant coefficient differential operator and the unique solution to an initial-value problem. Using this function, we generate a space of $\mathscr{D}$-polynomials and then construct a basis for this space. In Section 3 we introduce the $\mathscr{D}$-polar form for a $\mathscr{D}$-polynomial and in Section 4 we develop the Bernstein–Bézier theory for these polynomials. The spline function theory is developed in Section 5, and in Section 6 we consider knot insertion algorithms. Subdivision algorithms are considered in Section 7, and we complete the article by presenting an illustrative example.

2. $\mathscr{D}$-Polynomials

Throughout the paper $\mathscr{D}$ will denote a fixed two-dimensional space of real-valued functions which is the null space of an operator L of the form

$$L := D^2 + \gamma D + \delta, \qquad \gamma, \delta \in \mathbb{R}. \tag{2.1}$$

Moreover, $d \in \mathscr{D}$ will be the unique solution of the intial-value problem

$$Ld = 0, \qquad d(0) = 0, \qquad Dd(0) = 1.$$

It follows that $\mathscr{D} = \operatorname{span}\{d(\cdot - t), t \in \mathbb{R}\}$, the finite linear span of the set $\{d(\cdot - t),\ t \in \mathbb{R}\}$. Hence $\mathscr{D}$ is *translation invariant* i.e., if $f(\cdot) \in \mathscr{D}$ then for every $t \in \mathbb{R}$, $f(\cdot - t) \in \mathscr{D}$. In fact, any translation invariant two-dimensional space of continuous real-valued functions must be the null space of a differential operator of the form (2.1) [10].

Next, let

$$\mathscr{D}_n := \operatorname{span}\{d^n(\cdot - t), t \in \mathbb{R}\} = \operatorname{span}\{g^n, g \in \mathscr{D}\}, \qquad n \geqslant 0.$$

We will call elements of this space *$\mathscr{D}$-polynomials of degree n*. Clearly the space $\mathscr{D}_0$ consists of constant functions and $\mathscr{D}_1 = \mathscr{D}$. In general, the spaces $\mathscr{D}_n$ are not nested (i.e., $\mathscr{D}_n \not\subset \mathscr{D}_{n+1}$).

Remark 2.1 (Exponential polynomials with equidistant exponents). Let $L=(D-\mu)(D-\nu)$, $\mu, \nu \in \mathbb{C}$. By the definition of $\mathcal{D}_n$, every function in this space can be expressed as a (complex) linear combination of the functions

$$e^{[(n-i)\mu+i\nu]x}, \qquad i=0, ..., n,$$

if $\mu \neq \nu$, or the functions

$$x^i e^{n\mu x}, \qquad i=0, ..., n,$$

if $\mu=\nu$. In particular,

$$d(x)=\begin{cases}(e^{\mu x}-e^{\nu x})/(\mu-\nu), & \mu \neq \nu\\ xe^{\mu x}, & \mu=\nu.\end{cases}$$

Therefore the exponents form an arithmetic progression with the difference equal to $(\nu-\mu)x$. Thus $\mathcal{D}$-polynomials can be viewed as *exponential polynomials with equidistant exponents*. Such functions have also been considered in [13].

Remark 2.2 (The symmetric case $\gamma=0$). If $\gamma=0$ i.e., L is *self-adjoint*, then $\mathcal{D}$ is *symmetric* in the sense that $f(\cdot) \in \mathcal{D}$ implies $f(-\cdot) \in \mathcal{D}$. In this case the function d is given by

$$d(x)=\begin{cases}\sin(\sqrt{\delta}\, x)/\sqrt{\delta}, & \delta>0\\ x, & \delta=0\\ \sinh(\sqrt{-\delta}\, x)/\sqrt{-\delta}, & \delta<0,\end{cases}$$

which is an *antisymmetric* function (i.e., $d(x)=-d(-x)$). If, in addition $\delta=0$ then $\mathcal{D}_n=\Pi_n$, $n \geqslant 0$, where Π_n is the space of *algebraic polynomials of order* $n+1$ (or *degree* $\leqslant n$). On the other hand, if $\delta=1$ then $\mathcal{D}_n=\mathcal{T}_n$, where

$$\mathcal{T}_n:=\begin{cases}\operatorname{span}\{1, \sin(2x), \cos(2x), \sin(4x), \cos(4x), ..., \sin(nx), \cos(nx)\},\\ \qquad n \text{ even},\\ \operatorname{spn}\{\sin(x), \cos(x), \sin(3x), \cos(3x), ..., \sin(nx), \cos(nx)\},\\ \qquad n \text{ odd},\end{cases}$$

is the usual space of *trigonometric polynomials of order* $n+1$.

We will end this section by constructing a basis for $\mathcal{D}_n$. Let $a, b \in \mathbb{R}$ be such that $a<b$ and $d(b-a) \neq 0$ (and therefore also $d(a-b) \neq 0$). Moreover, let

$$b_0(x):=\frac{d(x-b)}{d(a-b)}, \; b_1(x):=\frac{d(x-a)}{d(b-a)}, \qquad x \in \mathbb{R}. \tag{2.2}$$

Since $\mathscr{D}$ is translation invariant, the functions b_0 and b_1 both belong to $\mathscr{D}$. In addition, they are linearly independent since

$$b_0(a)=1, \quad b_0(b)=0, \quad b_1(a)=0, \quad b_1(b)=1. \tag{2.3}$$

THEOREM 2.3 (Basis for $\mathscr{D}_n$). *The functions*

$$B_i^n(x) := \binom{n}{i} b_0^{n-i}(x)\, b_1^i(x), \qquad x \in \mathbb{R}, \quad i=0, ..., n,$$

form a basis for $\mathscr{D}_n$.

Proof. From the definition of $\mathscr{D}_n$ and the fact that $\dim \mathscr{D}=2$, it is clear that $\dim \mathscr{D}_n \leqslant n+1$. Thus it will be sufficient to show that the $B_i^n \in \mathscr{D}_n$, $i=0, ..., n$, are linearly independent. Proceeding by induction on n, the assertion is true for $n=0, 1$. For $n>1$, let

$$\sum_{i=0}^{n} c_i B_i^n(x)=0, \qquad \text{for all} \quad x \in \mathbb{R}.$$

In particular, this equality must hold for $x=b$, which gives $c_n=0$. However, the remaining sum is now a product of the function b_0 with a linear combination of the functions B_i^{n-1}, $i=0, ..., n-1$, which are linearly independent by the induction hypothesis. Hence the remaining coefficients c_i, $i=0, ..., n-1$, must also be zero. ∎

3. POLAR FORMS

In this section we recall the definition of a polar form for functions in $\mathscr{D}_n$ [10].

THEOREM 3.1 (Polar form). *For every $F \in \mathscr{D}_n$, $n \geqslant 0$, there exists a unique function $f(x_1, ..., x_n)$ of n variables, called a $\mathscr{D}$-polar form of F, satisfying the following properties*:

(a) *f is symmetric with respect to $x_1, ..., x_n$,*

(b) *f is equal to F on the diagonal i.e., $f(x, ..., x)=F(x)$, for all $x \in \mathbb{R}$,*

(c) *for all $m \geqslant 1$ and all real numbers $y, y_1, ..., y_m$, the function f is $\mathscr{D}$-affine i.e., f satisfies in each variable the relation*

$$f(\ldots, y, \ldots)=\sum_{i=1}^{m} \lambda_i\, f(\ldots, y_i, \ldots), \tag{3.1}$$

whenever the numbers $\lambda_1, ..., \lambda_m$ *are chosen so that*

$$g(y) = \sum_{i=1}^{m} \lambda_i g(y_i), \qquad \textit{for all} \quad g \in \mathscr{D}. \tag{3.2}$$

Proof. A polar form b_i^n of B_i^n is given by

$$b_i^n(x_1, ..., x_n) = \frac{\binom{n}{i}}{n!} \sum_{\pi} b_0(x_{\pi(1)}) \cdots b_0(x_{\pi(n-i)})\, b_1(x_{\pi(n-i+1)}) \cdots b_1(x_{\pi(n)}), \tag{3.3}$$

where the sum in (3.3) is taken over all permutations $\pi\colon \{1, ..., n\} \to \{1, ..., n\}$. This is easily proved by verifying all three defining properties (a)–(c). Next, since the functions B_i^n, $i = 0, ..., n$, form a basis for $\mathscr{D}_n$, we conclude that the function

$$f(x_1, ..., x_n) = \sum_{i=0}^{n} c_i b_i^n(x_1, ..., x_n), \qquad c_0, ..., c_n \in \mathbb{R},$$

is a polar form of the function $F \in \mathscr{D}_n$, given by

$$F(x) = \sum_{i=0}^{n} c_i b_i^n(x, ..., x) = \sum_{i=0}^{n} c_i B_i^n(x).$$

As for the uniqueness of this representation, it suffices to notice that the n-variate functions $b_i^n(x_1, ..., x_n)$, $i = 0, ..., n$, are linearly independent since they are linearly independent on the diagonal $x_1 = \cdots = x_n = x$. ∎

In the following we prove that (3.1) may be replaced by a three-term "recurrence" relation.

THEOREM 3.2 (Three-term recurrence relation). *A function f is* $\mathscr{D}$*-affine if and only if it satisfies the relation*

$$f(\ldots, y, \ldots) = \frac{d(y - y_2)}{d(y_1 - y_2)} f(\ldots, y_1, \ldots) + \frac{d(y - y_1)}{d(y_2 - y_1)} f(\ldots, y_2, \ldots), \tag{3.4}$$

for all $y, y_1, y_2 \in \mathbb{R}$, *such that* $d(y_2 - y_1) \neq 0$.

Proof. The proof of (3.4) is immediate by setting $m = 2$ in (3.1) and by observing that $d(y_2 - y_1) \neq 0$ implies

$$g(y) = \frac{d(y - y_2)}{d(y_1 - y_2)} g(y_1) + \frac{d(y - y_1)}{d(y_2 - y_1)} g(y_2), \tag{3.5}$$

for every $y \in \mathbb{R}$ and every $g \in \mathscr{D}$.

To prove the converse, fix $y, y_1, ..., y_m \in \mathbb{R}$ and let $\lambda_1, ..., \lambda_m$ satisfy (3.2). Moreover, suppose that $u_1, u_2 \in \mathbb{R}$ are such that $d(u_2 - u_1) \neq 0$. Then by (3.4),

$$\begin{aligned}
f(\ldots, y, \ldots) &= \frac{d(y-u_2)}{d(u_1-u_2)} f(\ldots, u_1, \ldots) \\
&\quad + \frac{d(y-u_1)}{d(u_2-u_1)} f(\ldots, u_2, \ldots) \\
&= \sum_{i=1}^{m} \lambda_i \left(\frac{d(y_i-u_2)}{d(u_1-u_2)} f(\ldots, u_1, \ldots) + \frac{d(y_i-u_1)}{d(u_2-u_1)} f(\ldots, u_2, \ldots) \right) \\
&= \sum_{i=1}^{m} \lambda_i \, f(\ldots, y_i, \ldots),
\end{aligned}$$

proving (3.1). ▮

A consequence of Theorem 3.2 is that $\mathscr{D}$-affinity means that in each variable the polar form of a function in $\mathscr{D}_n$ belongs to the space $\mathscr{D}$. We will denote functions in $\mathscr{D}_n$, $n \geqslant 2$, by capital letters and their polar forms by the corresponding small case letters. For $n = 1$, small case letters will be used since the polar form of a function from $\mathscr{D}_n$ is the function itself. From this point on, we will assume that $n \geqslant 1$ to avoid having to constantly list the exceptional and uninteresting case $n = 0$.

4. Bernstein–Bézier Representation

In this section we will develop a Bernstein–Bézier theory for the space $\mathscr{D}_n$. We call the functions B_i^n, $i = 0, ..., n$, defined in Theorem 2.3, the *Bernstein basis polynomials of degree n corresponding to $\mathscr{D}$* or simply *B-polynomials.* Clearly for $\mathscr{D}_n = \Pi_n$, these are the classical Bernstein basis polynomials whereas in the case $\mathscr{D}_n = \mathscr{T}_n$, we obtain the *circular* Bernstein basis polynomials considered in [1].

Theorem 4.1 (Polar form of B_i^n). *For $j = 0, ..., n$, let*

$$t_j := (\underbrace{a, ..., a}_{n-j}, \underbrace{b, ..., b}_{j}).$$

The polar form b_i^n of B_i^n satisfies

$$b_i^n(t_j) = \delta_{ij}, \qquad i, j = 0, ..., n.$$

Proof. By (2.3), the summands in (3.3) are nonzero for $(x_1, ..., x_n) = t_j$ only if $x_{\pi(1)} = \cdots = x_{\pi(n-i)} = a$, $x_{\pi(n-i+1)} = \cdots = x_{\pi(n)} = b$. Hence b_i^n equals zero unless $i = j$. In that case, the number of nonzero summands in (3.3) is $n!/\binom{n}{i}$ and so $b_i^n(t_j) = 1$. ▮

COROLLARY 4.2 (Dual basis for $\mathscr{D}_n$). *With t_j as in Theorem 4.1, the functionals $\mu_j\colon \mathscr{D}_n \to \mathbb{R}$, $j = 0, ..., n$, defined by*

$$\mu_j F := f(t_j),$$

where f is the polar form of $F \in \mathscr{D}_n$, form a dual basis for $\{B_i^n\}_{i=0}^n$ i.e.,

$$\mu_j B_i^n = \delta_{ij}, \qquad i, j = 0, ..., n.$$

From Corollary 4.2 it follows, as in the algebraic polynomial case, that the coefficients c_i, $i = 0, ..., n$, of a polynomial $F \in \mathscr{D}_n$ represented in the *B-form*

$$F(x) := \sum_{i=0}^{n} c_i B_i^n(x), \qquad x \in \mathbb{R}, \tag{4.1}$$

can be obtained by evaluating its polar form f at the points t_i i.e.,

$$c_i = f(t_i). \tag{4.2}$$

Next we generalize a result found in [15] for algebraic polynomial polar forms.

COROLLARY 4.3 (Polar interpolation). *Let $c_i \in \mathbb{R}$, $i = 0, ..., n$. There exists a unique function $F \in \mathscr{D}_n$ whose polar form f satisfies*

$$f(t_i) = c_i, \qquad i = 0, ..., n.$$

Proof. Let F be of the form (4.1) and let f be the polar form of F, then

$$\mu_i F = f(t_i) = c_i.$$

The uniqueness follows from the linear independence of the B-polynomials. ▮

It is also possible to define a Bernstein operator associated with the space $\mathscr{D}_n$. To that end, let

$$\xi_i := a + i\,\frac{b-a}{n}, \qquad i = 0, ..., n.$$

We define

$$\mathscr{L}_n := \operatorname{span}\{d(n\cdot - t), t \in \mathbb{R}\} = \operatorname{span}\{g(n\cdot), g \in \mathscr{D}\}. \tag{4.3}$$

In particular, $\mathscr{L}_1 = \mathscr{D}$. Similarly to $\mathscr{D}$, the space $\mathscr{L}_n$ is a two-dimensional translation invariant space. Unlike $\mathscr{D}$, however, $\mathscr{L}_n$ is a subspace of $\mathscr{D}_n$ for all n. As the next result shows, $\mathscr{L}_n$ can be considered as an analog of the space of linear functions.

COROLLARY 4.4 (Bernstein operator). *The operator*

$$B_n F(x) := \sum_{i=0}^{n} F(\xi_i)\, B_i^n(x),$$

defined on bounded real-valued functions on $\mathbb{R}$, *reproduces functions in* $\mathscr{L}_n$ *i.e.,*

$$B_n F \equiv F, \qquad \textit{for all} \quad F \in \mathscr{L}_n.$$

Proof. On account of Corollary 4.2, it suffices to prove that for all $F \in \mathscr{L}_n$,

$$f(t_i) = F(\xi_i), \qquad i = 0, ..., n. \tag{4.4}$$

Notice that the polar form f of a function $F \in \mathscr{L}_n$ is given by

$$f(x_1, ..., x_n) = F\left(\frac{\sum_{j=1}^{n} x_j}{n}\right). \tag{4.5}$$

This is because the translation invariance of $\mathscr{L}_n$ implies that the right hand side of (4.5) is, in each argument, a function in $\mathscr{D}$. By the defintion of the ξ_i, this readily proves (4.4). ∎

Remark 4.5. *B*-polynomials do not form a partition of unity unless $\mathscr{D}_n = \Pi_n$. Thus $B_n F$ will not necessarily converge to F as n goes to infinity, which is in contrast to the classical situation.

The following algorithm enables one to evaluate polar forms recursively.

ALGORITHM 4.6 (Evaluation algorithm for polar forms).

Let f be the polar form of a function $F \in \mathscr{D}_n$.

Set $c_i^0 := c_i$, $i = 0, ..., n$,

For $k = 1$ to n,

For $i = k$ to n, (4.6)

$$c_i^k := b_0(x_k)\, c_{i-1}^{k-1} + b_1(x_k)\, c_i^{k-1}.$$

Then $f(x_1, ..., x_n) = c_n^n$.

Proof. The assertion follows from the identity

$$c_i^k = f(x_1, ..., x_k, \underbrace{a, ..., a}_{n-i}, \underbrace{b, ..., b}_{i-k}),$$

which is a consequence of (3.4) applied to $y_1 = a$, $y_2 = b$. ▮

Remark 4.7. (a) By setting $x_1 = \cdots = x_n = x$ in the above algorithm we obtain an analog of the *de Casteljau algorithm* for evaluating polynomials in the B-form at a point x.

(b) Algorithm 4.6 can also be utilized to *convert* polynomials in the B-form from one interval to another. In particular, let c_i, $\bar{c}_i$, $i = 0, ..., n$, be the coefficients of a polynomial $F \in \mathscr{D}_n$ associated with the intervals $[a, b]$ and $[\bar{a}, \bar{b}]$ respectively. Thus by (4.2),

$$\bar{c}_i = f(\underbrace{\bar{a}, ..., \bar{a}}_{n-i}, \underbrace{\bar{b}, ..., \bar{b}}_{i}),$$

and hence Algorithm 4.6 can be applied with $x_1 = \cdots = x_{n-i} = \bar{a}$, $x_{n-i+1} = \cdots = x_n = \bar{b}$.

(c) Specializing (b) to the intervals $[a, b]$, $[a, s]$ and the intervals $[a, b]$, $[s, b]$ leads to a *subdivision* algorithm for a function $F \in \mathscr{D}_n$. Alternatively, the coefficients associated with the refined polynomials on the two intervals $[a, s]$ and $[s, b]$ can be read off of the array produced by the de Casteljau algorithm for evaluating F at the point s (in the same way as for algebraic polynomials [5]).

In order to consider *piecewise* polynomials, it is of interest to obtain conditions on a smooth join of two polynomials at a single point. Our next result gives these conditions in terms of polar forms of the respective polynomials.

THEOREM 4.8 (Contact of order m of two polynomials). *Two polynomials $F, G \in \mathscr{D}_n$ have contact of order $m \leqslant n$ at a point $s \in \mathbb{R}$, i.e.,*

$$D^k F(s) = D^k G(s), \qquad k = 0, ..., m,$$

if and only if their respective polar forms f and g satisfy

$$f(y_1, ..., y_m, s, ..., s) = g(y_1, ..., y_m, s, ..., s),$$

for all $y_1, ..., y_m \in \mathbb{R}$.

Proof. Without loss of generality, we may assume $G \equiv 0$ and $s = 0$. Let $F \in \mathscr{D}_n$ be a function with the property

$$D^k F(0) = 0, \qquad k = 0, ..., m.$$

In case $m=n$, these conditions imply $F \equiv 0$ and hence the statement of the theorem is trivially true. Therefore let $m<n$. By using representation (4.1), it is not difficult to prove e.g., by induction on m, that F has the form

$$F(x)=d^{m+1}(x)\,H(x),$$

where $H \in \mathscr{D}_{n-m-1}$. Thus f is given by

$$f(x_1, ..., x_n)=\frac{1}{n!}\sum_{\pi} d(x_{\pi(1)})\cdots d(x_{\pi(m+1)})\,h(x_{\pi(m+2)}, ..., x_{\pi(n)}), \tag{4.7}$$

where the sum in (4.7) is taken over all permutations π: $\{1, ..., n\} \to \{1, ..., n\}$. If $(x_1, ..., x_n)=(y_1, ..., y_m, 0, ..., 0)$ then for any π, at least one of the variables $x_{\pi(1)} \cdots x_{\pi(m+1)}$ is zero and thus the sum on the right-hand side of (4.7) vanishes, proving that

$$f(y_1, ..., y_m, 0, ..., 0)=0, \qquad \text{for all} \quad y_1, ..., y_m \in \mathbb{R}. \tag{4.8}$$

The opposite direction of the assertion of the theorem, while being trivial for $m=n$, can be proved for $m<n$ by showing e.g., again by induction on m, that any polar form f satisfying (4.8) has the representation (4.7). Moreover, by the chain rule, we observe

$$D^k F(x)=\frac{d^k}{dx^k} f(x, ..., x)=\sum_{u_1, ..., u_k \in \{x_1, ..., x_n\}} \frac{\partial^k}{\partial u_1 \cdots \partial u_k} f(x, ..., x).$$

Differentiating each term in (4.7) partially with respect to $u_1, ..., u_k$, $k \leqslant m$, leads to a sum of functions each of which is a multiple of $d(x_i)$ for some $i \in \{1, ..., n\}$. Thus for $x_1= \cdots =x_n=x=0$, each summand vanishes and the proof is complete. ∎

The following assertion is an analog of the well-known *smoothness condition* for the classical case of Bernstein–Bézier curves. It is a straightforward consequence of Theorem 4.8, (4.2), and the recurrence relation (3.4), and can be proved by induction on m.

COROLLARY 4.9. (C^m-continuity conditions for polynomials in the B-form). *Let F and G be polynomials defined on the intervals $[a, s]$ and $[s, b]$ with coefficients c_i and d_i, $i=0, ..., n$, respectively. Then F and G have contact of order m at the point s if and only if*

$$d_i=\sum_{j=0}^{i} c_{n-i+j} B_j^i(b), \qquad i=0, ..., m,$$

where B_j^i are the B-polynomials with respect to the interval $[a, s]$.

Remark 4.10. It is well known that two polynomials join with C^1-continuity if and only if three neighboring control points all lie on a line. In our more general setting, using Corollary 4.9, one can prove a similar result under the condition that the three control points (see the definition below) lie on a curve of the form $(x, g(x))$, $g \in \mathscr{L}_n$. This has been proved for $\mathscr{D}_n = \mathscr{T}_n$ in [1].

Corollary 4.4 motivates the following

DEFINITION 4.11 (Control points and control curve). Let F be defined as in (4.1). The points $C_i := (\xi_i, c_i)$, $i = 0, ..., n$, will be called the *control points* of F. Let g_i, $i = 1, ..., n$, be the unique functions from $\mathscr{L}_n$, which interpolate the control points C_{i-1}, C_i; more precisely, define

$$g_i(t) := \frac{d(n(t-\xi_i))}{d(n(\xi_{i-1}-\xi_i))} c_{i-1} + \frac{d(n(t-\xi_{i-1}))}{d(n(\xi_i-\xi_{i-1}))} c_i, \qquad t \in [\xi_{i-1}, \xi_i].$$

We call the curve C consisting of the pieces $G_i := \{(t, g_i(t)), t \in [\xi_{i-1}, \xi_i]\}$, $i = 1, ..., n$, the *control curve* of F.

We are now ready to give a *geometric interpretation* of the Casteljau algorithm for polynomials in the B-form. Suppose c_i^k are the numbers produced by the de Casteljau algorithm for a point $x \in [a, b]$. For each $k = 0, ..., n$, and $i = k, ..., n$, let

$$C_i^k := (\xi_i^k, c_i^k),$$

where

$$\xi_i^k := \frac{kx + (n-i)\, a + (i-k) b}{n}. \tag{4.9}$$

Thus in particular $\xi_i^0 := \xi_i$, $i = 0, ..., n$.

PROPOSITION 4.12 (Geometric interpretation of the de Casteljau algorithm). *For each $k = 1, ..., n$, and $i = k, ..., n$, let*

$$G_i^k(t) := (t, g_i^k(t)), \qquad t \in [\xi_{i-1}^{k-1}, \xi_i^{k-1}],$$

where g_i^k is the unique function in $\mathscr{L}_n$ which interpolates c_{i-1}^{k-1} and c_i^{k-1} at ξ_{i-1}^{k-1} and ξ_i^{k-1}, respectively. Then

$$C_i^k = G_i^k(\xi_i^k), \qquad k = 1, ..., n, \qquad i = k, ..., n.$$

In particular,

$$C_n^n = G_n^n(x).$$

Proof. The function g_i^k is given by

$$g_i^k(t) = \frac{d(n(t-\xi_i^{k-1}))}{d(n(\xi_{i-1}^{k-1}-\xi_i^{k-1}))}\, c_{i-1}^{k-1} + \frac{d(n(t-\xi_{i-1}^{k-1}))}{d(n(\xi_i^{k-1}-\xi_{i-1}^{k-1}))}\, c_i^{k-1}.$$

With $t=\xi_i^k$, this reduces to formula (4.6) and we have $g_i^k(\xi_i^k)=c_i^k$. For $k=i=n$, obviously $\xi_i^k=x$. ∎

5. Splines

In this section we introduce spaces of piecewise polynomials, i.e., piecewise functions whose pieces are elements of $\mathscr{D}_n$. Let z be a positive number which satisfies the condition that if $|x-y|<z$ and $d(x-y)=0$ then $x=y$. Such a number exists because the function d either has periodic zeros or satisfies $d(x)=0$ if and only if $x=0$ (cf. Remark 2.2). Let

$$X := \{x_0 \leqslant \cdots \leqslant x_{n+q+1}\} \subset \mathbb{R}, \qquad q \geqslant n,$$

be a collection of knots, called a *knotvector*, satisfying

$$a = x_0 = \cdots = x_n, \qquad x_{q+1} = \cdots = x_{n+q+1} = b$$

and

$$0 < x_{i+n+1} - x_i < z, \qquad i = 0, ..., q.$$

We shall be concerned with the spaces $\mathscr{D}_{n,X}$ of *splines of degree n associated with the space $\mathscr{D}$ and the knotvector X*, defined by

$$\mathscr{D}_{n,X} := \operatorname{span}\{B_i^n\}_{i=0}^q,$$

where B_i^n is the (normalized) *B-spline of degree n*, defined inductively by

$$B_i^0(x) := \begin{cases} 1, & x_i \leqslant x < x_{i+1} \\ 0, & \text{otherwise}, \end{cases}$$

and for $k=1, ..., n$, by

$$B_i^k(x) := \frac{d(x-x_i)}{d(x_{i+k}-x_i)}\, B_i^{k-1}(x) + \frac{d(x-x_{i+k-1})}{d(x_{i+1}-x_{i+k+1})}\, B_{i+1}^{k-1}(x). \tag{5.1}$$

If $d(x_{i+k-1}-x_i)=0$ and/or $d(x_{i+1}-x_{i+k+1})=0$, then the corresponding undefined terms in (5.1) should be set equal to zero. We will maintain this convention throughout the remainder of the paper. Since the B-splines reduce to B-polynomials in the case $q=n$, we have employed the same notation for both types of functions. In fact, all results of this and the subsequent sections subsume the results of Section 4 in the case $q=n$.

It follows from this definition that the B-spline B_i^n is a locally supported function with support $[x_i, x_{i+n+1}]$. Moreover, the results of this section imply that for distinct knots, the B-spline is an $n-1$ times continuously differentiable function. The recurrence relation (5.1) reduces to the classical recursion for polynomials, trigonometric, and hyperbolic B-splines when $\mathscr{D}_n$ is equal to Π_n, $\mathscr{T}_n$, or to the space of hyperbolic functions of degree n, respectively [17, 19]. For a general space $\mathscr{D}_n$, the relation (5.1) seems to be new. Notice the term $d(x-x_{i+k+1})/d(x_{i+1}-x_{i+k+1})$ in (5.1) which can be replaced by the more familiar term $d(x_{i+k+1}-x)/d(x_{i+k+1}-x_{i+1})$ only in the symmetric case (cf. Remark 2.2).

Remark 5.1. A more traditional way of defining the space $\mathscr{D}_{n,X}$ is to consider the space of functions which belong locally to $\mathscr{D}_n$ and which have prescribed degree of continuity at the knots. Using a standard approach (see e.g., [17]) one can show that this definition and the one given above are equivalent.

Henceforth, let $B_{i,j}^n$ be the polynomial from $\mathscr{D}_n$, which agrees with the B-spline B_i^n, on the interval $[x_j, x_{j+1})$.

LEMMA 5.2 (Recursion for the polar form of $B_{i,j}^n$). *The polar form $b_{i,j}^n$ of $B_{i,j}^n$ can be computed recursively by*

$$b_{i,j}^0 = \delta_{i,j} \tag{5.2}$$

$$\begin{aligned} b_{i,j}^k(y_1, ..., y_k) = {} & \frac{d(y_k - x_i)}{d(x_{i+k} - x_i)}\, b_{i,j}^{k-1}(y_1, ..., y_{k-1}) \\ & + \frac{d(y_k - x_{i+k+1})}{d(x_{i+1} - x_{i+k+1})}\, b_{i+1,j}^{k-1}(y_1, ..., y_{k-1}), \\ & k = 1, ..., n. \end{aligned} \tag{5.3}$$

Proof. If we set $y_i = x$ where $x_j \leqslant x < x_{j+1}$, then equation (5.3) is the same as equation (5.1). Induction can be used to show that in each variable $b_{i,j}^k$ is in $\mathscr{D}$ and furthermore that $b_{i,j}^k$ is symmetric in the variables $y_1, ..., y_{k-1}$. To show that $b_{i,j}^k$ is also symmetric in y_{k-1} and y_k, first apply (5.3) to each term on the right hand side of (5.3), then check the symmetry for the two cases, $d(x) = xe^{\mu x}$ and $d(x) = (e^{\mu x} - e^{\nu x})/(\mu - \nu)$ (cf. Remark 2.1). ▮

THEOREM 5.3 (Polar form of $B^n_{i,j}$). *For $x_j < x_{j+1}$ and*

$$j - n \leqslant k \leqslant j, \tag{5.4}$$

the polar form $b^n_{i,j}$ evaluated at the knots $x_{k+1}, ..., x_{k+n}$, satisfies

$$b^n_{i,j}(x_{k+1}, ..., x_{k+n}) = \delta_{i,k}. \tag{5.5}$$

Proof. We proceed inductively on n. The case $n=0$ is just a restatement of (5.2). If (5.5) is true for $n-1$ and $k > j-n$ then a direct application of (5.3) gives the result. If $k = j-n$, use (5.3) to eliminate x_{k+1}. Thus after rearranging terms we obtain

$$b^n_{i,j}(x_{k+1}, ..., x_{k+n}) = \frac{d(x_{k+1} - x_i)}{d(x_{i+n} - x_i)}\, b^{n-1}_{i,j}(x_{k+2}, ..., x_{k+n}) + \frac{d(x_{k+1} - x_{i+n+1})}{d(x_{i+1} - x_{i+n+1})}\, b^{n-1}_{i+1,j}(x_{k+2}, ..., x_{k+n}).$$

Applying the induction hypothesis completes the proof. ∎

We now use Theorem 5.3 to prove that (5.1) produces smooth functions.

COROLLARY 5.4 (Smoothness of the B-spline). *If $x_j < x_{j+1} = \cdots = x_{j+n-m} < x_{j+n-m+1}$ is a knot of multiplicity $n-m > 0$, then $B^n_{i,j}$ and $B^n_{i,j+n-m}$ have contact of order m at the point $s = x_{j+1} = \cdots = x_{j+n-m}$. That is, B^n_i is C^m-continuous at s.*

Proof. Applying Theorem 5.3 to the polar forms of $B^n_{i,j}$ and $B^n_{i,j+n-m}$ gives

$$b^{n-m}_{i,k}(s, ..., s) = b^{n-m}_{i,k}(x_{j+1}, ..., x_{j+n-m}) = \delta_{ij} \qquad \text{for} \quad k = j, j+n-m.$$

Next, by Lemma 5.2 we have

$$b^n_{i,j}(y_1, ..., y_m, s \cdots, s) = b^n_{i,j+n-m}(y_1, ..., y_m, s, ..., s).$$

Therefore by Theorem 4.8 we have the desired result. ∎

Next, we turn our attention to spline series of the type

$$F(x) := \sum_{i=0}^{q} c_i B^n_i(x). \tag{5.6}$$

The coefficients $c_i \in \mathbb{R}$ can be expressed in terms of the polar form associated with F. Let f_j be the polar form of the polynomial F_j, which coincides with F on the interval $[x_j, x_{j+1})$.

COROLLARY 5.5 (Dual basis for $\mathscr{D}_{n,X}$). *For* $j=0, ..., q$, *the functionals* $\mu_j\colon \mathscr{D}_{n,X} \to \mathbb{R}$, *defined by*

$$\mu_j F := f_j(x_{j+1}, ..., x_{j+n}), \qquad (= \cdots = f_{j+n}(x_{j+1}, ..., x_{j+n}))$$

form a dual basis for $\{B_i^n\}_{i=0}^q$ *i.e.*,

$$\mu_j B_i^n = \delta_{ij}, \qquad i, j = 0, ..., q.$$

COROLLARY 5.6 (Spline coefficients in terms of polar forms). *Let* $x_j < x_{j+1}$ *and let* i *be such that* $j-n \leqslant i \leqslant j$. *Then the coefficients* c_i *in* (5.6) *can be computed as*

$$c_i = f_j(x_{i+1}, ..., x_{i+n}). \tag{5.7}$$

Proof. Since $F_j = \sum_{k=j-n}^{j} c_k B_{k,j}^n$, we have by (5.5)

$$f_j(x_{i+1}, ..., x_{i+n}) = \sum_{k=j-n}^{j} c_k b_{k,j}^n(x_{i+1}, ..., x_{i+n}) = c_i. \quad \blacksquare$$

We now provide a version of the well-known algorithm for computing values of the spline F.

ALGORITHM 5.7 (Evaluation algorithm for splines).

Let m be such that $x_m \leqslant x < x_{m+1}$.

Set $c_i^0 := c_i$, $i = m-n, ..., m$,
For $k = 1$ to n,
For $i = m-n+k$ to m,

$$c_i^k := \frac{d(x - x_{i+n+1-k})}{d(x_i - x_{i+n+1-k})}\, c_{i-1}^{k-1} + \frac{d(x - x_i)}{d(x_{i+n+1-k} - x_i)}\, c_i^{k-1}.$$

Then $F(x) = c_m^n$.

Proof. As a consequence of Theorem 5.3, the symmetry and affinity of f_m and (3.5), we have

$$c_i^k = f_m(x_{i+1}, ..., x_m, \underbrace{x, ..., x}_{k}, x_{m+1}, ..., x_{i+n+1}),$$

which in turn gives

$$c_m^n = f_m(x, ..., x) = F_m(x) = F(x). \quad \blacksquare$$

A simple consequence of Corollary 5.6 is

COROLLARY 5.8 (Marsden identity). *For each $y \in \mathbb{R}$, we have*

$$d^n(y-x) = \sum_{i=0}^{q} d(y-x_{i+1}) \cdots d(y-x_{i+n})\, B_i^n(x), \qquad x \in \mathbb{R}.$$

Proof. It is easy to check that the polar form of the function $d^n(y-x)$ is $\prod_{j=1}^{n} d(y-x_j)$. The result is now an immediate consequence of Corollary 5.6. ∎

COROLLARY 5.9 (Schoenberg operator). *Let B_n be the operator, defined on bounded real-valued functions on $[a, b]$, by*

$$B_n F(x) := \sum_{i=0}^{q} F(\xi_i)\, B_i^n(x),$$

where ξ_i are knot averages given by

$$\xi_i := \frac{1}{n} \sum_{j=i+1}^{i+n} x_j. \tag{5.8}$$

Then B_n reproduces the space $\mathscr{L}_n$ (see (4.3)).

Proof. Using (4.5) and then Corollary 5.6, we get

$$\begin{aligned} B_n F(x) &= \sum_{i=0}^{q} F(\xi_i)\, B_i^n(x) = \sum_{i=0}^{q} f(x_{i+1}, ..., x_{i+n})\, B_i^n(x) \\ &= \sum_{i=0}^{q} f_i(x_{i+1}, ..., x_{i+n})\, B_i^n(x) \\ &= \sum_{i=0}^{q} c_i B_i^n(x) = F(x). \quad ∎ \end{aligned}$$

The operator B_n is an analog of the classical Schoenberg operator. For $\mathscr{D}_n = \mathscr{T}_n$, this operator has been introduced in [11]. It follows from Corollary 5.9 that the location of the points ξ_i is the same as for the classical polynomial splines. On the other hand, Corollary 5.9 and the results of Section 4 suggest that the notion of a control polygon as a piecewise linear function is no longer appropriate. Namely, the "control polygon" should be defined as the function interpolating the points (ξ_i, c_i) and which belongs piecewisely to the space $\mathscr{L}_n$.

DEFINITION 5.10 (Control Points and control curve). Let F be a spline of the form (5.6). The points $C_i := (\xi_i, c_i)$, $i = 0, ..., q$, will be called the *control points* of F. The function C which interpolates the values c_i at the points ξ_i, $i=0, ..., q$, and which is such that $C|_{(\xi_{i-1}, \xi_i)} \in \mathscr{L}_n$, $i=1, ..., q$, will be called the *control curve* of the spline F.

Based on the above definition it is possible to interpret Algorithm 5.7 along the same lines as in the case of polynomials in B-form in Section 4. For a more detailed treatment concerning the trigonometric case, see [12].

We conclude the section by giving an explicit representation for the control curve C in terms of the polar form of an associated polynomial from $\mathscr{D}_n$.

PROPOSITION 5.11 (Explicit representation for control curves). *Let* $x \in [a, b]$ *and* $1 \leqslant i \leqslant q$ *such that* $x \in [\xi_{i-1}, \xi_i]$. *Moreover, let* $i \leqslant j \leqslant i+n-1$. *Then*

$$C(x) = f_j(nx - x_{i+1} - \cdots - x_{i+n-1}, x_{i+1}, ..., x_{i+n-1}). \tag{5.9}$$

Proof. By definition of a control curve and by (5.7), C has the form

$$\begin{aligned} C(x) &= \frac{d(n(x-\xi_i))}{d(n(\xi_{i-1}-\xi_i))} c_{i-1} + \frac{d(n(x-\xi_{i-1}))}{d(n(\xi_i-\xi_{i-1}))} c_i \\ &= \frac{d(n(x-\xi_i))}{d(n(\xi_{i-1}-\xi_i))} f_j(x_i, ..., x_{i+n-1}) + \frac{d(n(x-\xi_{i-1}))}{d(n(\xi_i-\xi_{i-1}))} f_j(x_{i+1}, ..., x_{i+n}), \end{aligned}$$

which on account of (5.8) and (3.4), with $y_1 = x_i$, and $y_2 = x_{i+n}$, readily proves (5.9). ▮

Remark 5.12. It is possible to establish results analogous to the *convex hull property* and the *variation diminishing property* of algebraic polynomials. The case $\mathscr{D}_n = \mathscr{T}_n$ was considered in [12] (see also [1]) and the general case can be established along the same lines as in that paper.

6. KNOT INSERTION

In this section, $\bar{X}$ will denote a refined knotvector of X i.e., $X \subset \bar{X}$ and for the remainder of the paper we will use bars to designate quantities associated with $\bar{X}$. The spline $F(x)$ in (5.6) can be expressed on the finer knotvector $\bar{X}$ as

$$F(x) = \sum_{i=0}^{\bar{q}} \bar{c}_i \bar{B}_i^n(x),$$

since clearly $\mathscr{D}_{n,X}$ is a subspace of $\mathscr{D}_{n,\bar{X}}$ (see Remark 5.1). In order to represent the new coefficients $\bar{c}_i$ in terms of the original c_i, we need a representation of a B-spline $B_i^n(x)$ as

$$B_i^n(x) = \sum_{j=0}^{\bar{q}} \beta_{i,j}^n \bar{B}_j^n(x),$$

which, by (5.7) yields

$$\beta_{i,j}^n = b_{i,j}^n(\bar{x}_{j+1}, ..., \bar{x}_{j+n}).$$

This means that the coefficients $\bar{c}_j$ can be computed by

$$\bar{c}_j = \sum_{i=0}^{q} \beta_{i,j}^n c_i,$$

which, on account of Lemma 5.2 leads to the recurrence relation

$$\beta_{i,j}^k = \frac{d(\bar{x}_{j+k} - x_i)}{d(x_{i+k} - x_i)} \beta_{i,j}^{k-1} + \frac{d(\bar{x}_{j+k} - x_{i+k+1})}{d(x_{i+1} - x_{i+k+1})} \beta_{i+1,j}^{k-1}, \qquad k = 1, ..., n,$$

where

$$\beta_{i,j}^0 = \begin{cases} 1, & \text{for} \quad x_i \leqslant \bar{x}_j < x_{i+1} \\ 0, & \text{otherwise.} \end{cases}$$

The function $\beta_{i,j}^n$ is an analog of the familiar *discrete B-spline* ([6, 17]). The above recursions give rise to the Oslo algorithm for splines.

ALGORITHM 6.1 (Oslo algorithm).

Let $1 \leqslant i \leqslant \bar{q}$ and let m be such that $x_m \leqslant \bar{x}_i < x_{m+1}$.

Set $c_{j,i}^0 := c_j, j = k-n, ..., k,$

For $k = 1$ to n,

For $j = m - n + k$ to m,

$$c_{j,i}^k := \frac{d(\bar{x}_{i+n+1-k} - x_{j+n+1-k})}{d(x_j - x_{j+n+1-k})} c_{j-1,i}^{k-1} + \frac{d(\bar{x}_{i+n+1-k} - x_j)}{d(x_{j+n+1-k} - x_j)} c_{j,i}^{k-1}.$$

Then $\bar{c}_i = c_{m,i}^n$.

In the case of inserting one knot at a time, similar identities can be obtained as in the polynomial case [2]. For the sake of convenience of the reader we include the main formulae. These results are given for trigonometric splines in [12].

THEOREM 6.2 (Recurrence relation for the new coefficients). *Let $\bar{X}=(a, ..., x_m, x, x_{m+1}, ..., b)$ be a refined knotvector formed by adding one additional knot to X. The new coefficients $\bar{c}_i$ can be computed using the following*:

$$\bar{c}_i=\begin{cases} c_i, & i\leqslant m-n \\ \dfrac{d(x-x_{i+n})}{d(x_i-x_{i+n})}\,c_{i-1}+\dfrac{d(x-x_i)}{d(x_{i+n}-x_i)}\,c_i, & m-n+1\leqslant i\leqslant m \\ c_{i-1}, & m+1\leqslant i. \end{cases}$$

Proof. By appling Corollary 5.6 to $\bar{X}$, the new coefficients $\bar{c}_i$ can be determined as

$$\bar{c}_i=\begin{cases} f_j(x_{i+1}, ..., x_{i+n}), \\ \qquad i\leqslant m-n,\ i\leqslant j\leqslant i+n \\ f_m(x_{i+1}, ..., x_m, x, x_{m+1}, ..., x_{i+n-1}), \\ \qquad m-n+1\leqslant i\leqslant m \\ f_j(x_i, ..., x_{i+n-1}), \\ \qquad m+1\leqslant i,\ i\leqslant j+1\leqslant i+n. \end{cases}$$

With these identities, the algorithm can easily be established. ∎

We finish this section by noting that, as in the polynomial case, by inserting multiple knots into the spline curve such that every knot has multiplicity $n+1$, the spline can be converted into a piecewise curve whose individual pieces are represented in the B-form.

7. SUBDIVISION

In accordance with commonly accepted terminology, by *subdivision* we mean a representation of a spline function in terms of a *refined basis*. The control points of the spline corresponding to that basis typically converge to the spline as the number of refinement steps increases [5]. Knot insertion serves as a natural method for subdividing splines. As more knots are inserted into the spline, the refined control curves converge to the original smooth spline curve. In this section we prove that the convergence is quadratic. We first need an auxiliary lemma.

LEMMA 7.1 (Estimate for polar forms). *Let $[u, v]$ be an interval containing the points $y_1, ..., y_n \in \mathbb{R}$. Then for every $G \in \mathcal{D}_n$ there exists a constant K depending on n, G and $[u, v]$, but not on $y_1, ..., y_n$, such that*

$$|g(y_1, ..., y_n) - G(y)| \leqslant Kh^2,$$

where $y := (y_1 + \cdots + y_n)/n$ and $h := \max_{1 \leqslant i, j \leqslant n}\{|y_i - y_j|\}$.

Proof. Expressing g in terms of a Taylor polynomial centered at $(y, ..., y)$ leads to

$$g(y_1, ..., y_n) - g(y, ..., y) = \sum_{i=1}^{n} \frac{\partial g(y, ..., y)}{\partial y_i}(y_i - y) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 g(\eta_1, ..., \eta_n)}{\partial y_i \partial y_j}(y_i - y)(y_j - y),$$

where $\eta_i = ty + (1-t)\,y_i$, $i = 1, ..., n$, for some $t \in (0, 1)$. The first order term on the right-hand side of the equation is zero since by symmetry of g, the partial derivatives of g evaluated at $(y, ..., y)$ are all equal, and since $(y_1 - y) + \cdots + (y_n - y) = 0$. Setting

$$K := \frac{n^2}{8} \sup_{\substack{\eta_1, ..., \eta_n \in [u, v], \\ 1 \leqslant i, j \leqslant n}} \left\{\left|\frac{\partial^2 g}{\partial y_i \partial y_j}(\eta_1, ..., \eta_n)\right|\right\},$$

and using the inequality

$$|(y_i - y)(y_j - y)| \leqslant \frac{h^2}{4},$$

completes the proof. ▮

THEOREM 7.2 (Convergence of the control curve). *Let $F \in \mathcal{D}_{n,X}$ be a spline of the form (5.6) and C be its control curve. Moreover, let $i = 1, ..., q$, and $x \in [\xi_{i-1}, \xi_i]$. Then*

$$|C(x) - F(x)| \leqslant K(x_{i+n} - x_i)^2, \tag{7.1}$$

where the constant K is the same as in Lemma 7.1, with $u = x_i$, $v = x_{i+n}$, and $G = F_j$, where j is such that $x \in [x_j, x_{j+1}]$.

Proof. By Proposition 5.11, we have

$$C(x) = f_j(nx - x_{i+1} - \cdots - x_{i+n-1}, x_{i+1}, ..., x_{i+n-1}).$$

Setting $G=F_j$, $g=f_j$ and $y_1=nx-x_{i+1}-\cdots-x_{i+n-1}$, $y_2=x_{i+1}, ..., y_n=x_{i+n-1}$, in Lemma 7.1, and observing that $y=(y_1+\cdots+y_n)/n=x$, proves (7.1), once we have shown

$$\max_{1\leqslant k,\, l\leqslant n} \{|y_k-y_l|\} \leqslant |x_{i+n}-x_i|.$$

However, this follows directly from the fact that $x\in[\xi_{i-1}, \xi_i]$ or, equivalently, $y_1\in[x_i, x_{i+n}]$. ∎

Remark 7.3. Theorem 7.2 generalizes the corresponding results obtained by different means in [7] and [8] for polynomial splines, and in [12] for trigonometric splines. As in [12], this theorem does not require any assumptions about the smoothness of the spline F.

8. Examples

In this section we illustrate the results of the preceding sections with a concrete example of a space $\mathscr{D}$, which is the null space of the operator

$$L:=D^2-3D+2.$$

Thus $\mathscr{D}=\operatorname{span}\{e^{2x}, e^x\}$ and $d(x)=e^{2x}-e^x$. The relations (2.2) specialize to

$$b_0(x)=\frac{e^{2(x-b)}-e^{x-b}}{e^{2(a-b)}-e^{a-b}}, \qquad b_1(x)=\frac{e^{2(x-a)}-e^{x-a}}{e^{2(b-a)}-e^{b-a}}.$$

For $[a, b]=[0, 1]$, the functions b_0 and b_1 are displayed in Fig. 1.

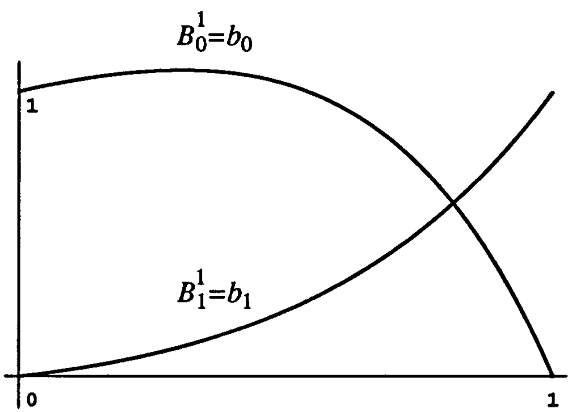

Fig. 1. The functions b_0 and b_1.

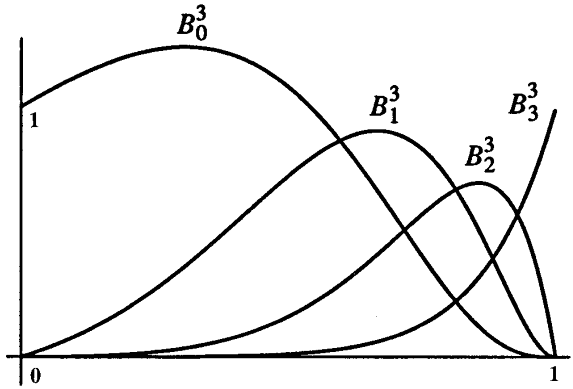

FIG. 2. The four cubic B-polynomials.

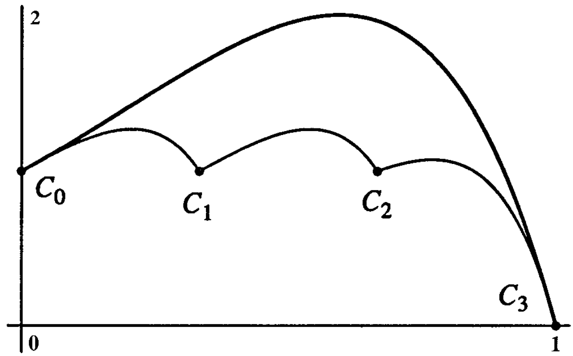

FIG. 3. A cubic polynomial together with its control curve.

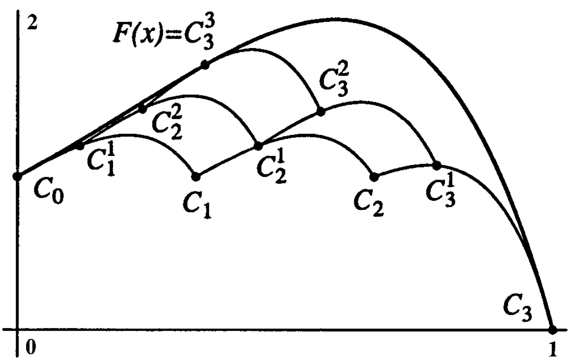

FIG. 4. The geometric interpretation of the de Casteljau algorithm.

The corresponding cubic B-polynomials ($n=3$) are depicted in Fig. 2. Figure 3 shows a typical linear combination of the B-polynomials together with its associated control points and control curve. The curve corresponds to coefficients $c_0=1$, $c_1=1$, $c_2=1$, $c_3=0$. The space $\mathcal{D}_3$, to which this function belongs, is the null space of the differential operator $(D-6)(D-5)(D-4)(D-3)$ and hence $\mathcal{D}_3=\operatorname{span}\{e^{6x}, e^{5x}, e^{4x}, e^{3x}\}$, a space of exponential polynomials with equidistant exponents.

Figure 4 illustrates the steps of the de Casteljau algorithm for the evaluation of the polynomial from Fig. 3 at $x=0.35$. For example, the coefficient c_2^1 is obtained as a $\mathcal{D}$-affine linear combination of c_1 and c_2, by

$$c_2^1=f(x,0,1)=b_0(x)\,f(0,0,1)+b_1(x)\,f(0,1,1)=b_0(x)\,c_1+b_1(x)\,c_2.$$

Similarly, c_3^3 is a $\mathcal{D}$-affine linear combination of c_2^2 and c_3^2:

$$\begin{aligned} F(x)=c_3^3=f(x,x,x)&=b_0(x)\,f(x,x,0)+b_1(x)\,f(x,x,1)\\ &=b_0(x)\,c_2^2+b_1(x)\,c_3^2. \end{aligned}$$

The abscissae corresponding to these coefficients are obtained from (4.9). Thus for example $\xi_2^1=0.45$, and so $C_2^1=(0.45, c_2^1)$.

Figure 5 shows a cubic B-spline with uniform knots 0, 1/2, 1, 3/2, 2, which is two times continuously differentiable (C^2). Note the asymmetry of the B-spline with respect to the midpoint of the interval $[0, 2]$ which is a consequence of the fact that the space $\mathcal{D}$ is not symmetric.

The geometric interpretation of the evaluation Algorithm 5.7 of a typical spline is illustrated in Fig. 6. Note the resemblance with the de Casteljau algorithm in Fig. 4. The parameters of the spline were chosen as follows: $n=3$, $x_0=\cdots=x_3=0$, $x_4=1/3$, $x_5=2/3$, $x_6=\cdots=x_9=1$, and $c_0=2$, $c_1=2$, $c_2=3$, $c_3=2$, $c_4=2$, $c_5=0$. Thus F is a cubic spline on the interval $[0, 1]$ with two interior knots. The evaluation algorithm is illustrated at the point $x=0.55$. Since $x\in[x_4, x_5)$, we have $F(x)=F_4(x)=f_4(x,x,x)=c_4^3$. The spline coefficients involved in the computation of the value $F(x)$ are given in terms of the polar form f_4 as

$$\begin{aligned} c_1&=f_4(x_1,x_2,x_3)=f_4(0,0,1/3), & c_2&=f_4(x_2,x_3,x_4)=f_4(0,1/3,2/3),\\ c_3&=f_4(x_3,x_4,x_5)=f_4(1/3,2/3,1), & c_4&=f_4(x_3,x_4,x_5)=f_4(2/3,1,1). \end{aligned}$$

For example, Algorithm 5.7 leads to

$$\begin{aligned} f_4(x_3,x_4,x)=c_2^1&=\frac{d(x-1/3)}{d(-1/3)}\,c_1+\frac{d(x)}{d(1/3)}\,c_2\\ &=\frac{d(x-x_4)}{d(x_1-x_4)}f_4(x_1,x_2,x_3)+\frac{d(x-x_1)}{d(x_4-x_1)}f_4(x_2,x_3,x_4) \end{aligned}$$

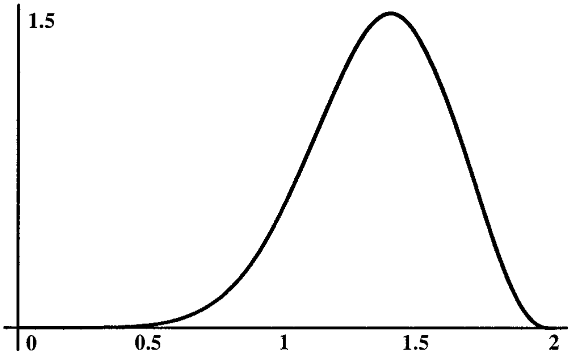

FIG. 5. A cubic B-spline on a uniform knotvector.

and

$$F(x) = f_4(x, x, x) = c_4^3 = \frac{d(x-2/3)}{d(-1/3)} c_3^2 + \frac{d(x-1/3)}{d(1/3)} c_4^2$$

$$= \frac{d(x-x_5)}{d(x_4-x_5)} f_4(x_4, x, x) + \frac{d(x-x_4)}{d(x_5-x_4)} f_4(x_5, x, x).$$

The abscissa corresponding to the coefficient c_2^1 is $\xi_2^1 = \frac{1}{3}(x_3 + x_4 + x) \approx .29$.

An example of knot insertion is given in Fig. 7. The knot $x = 0.55$ is inserted into the spline curve of Fig. 6 up to three times. As can be seen, inserting the knot x a total of three times gives the value of the spline at the point x. In fact, the control points in Fig. 6 are identical with the ones obtained by knot insertion. In particular, comparing the last figure in Fig. 7 with Fig. 6 gives $\bar{C}_1 = C_1$, $\bar{C}_2 = C_2^1$, $\bar{C}_3 = C_3^2$, $\bar{C}_4 = C_4^3$, $\bar{C}_5 = C_4^2$,

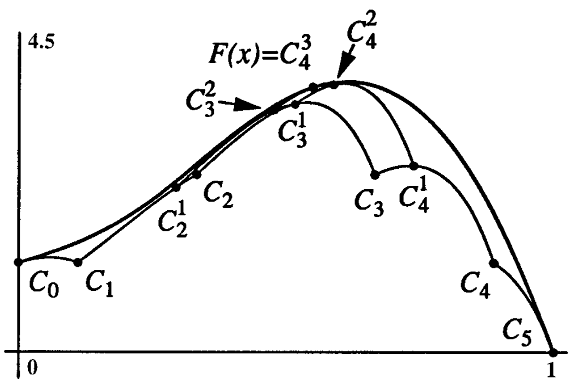

FIG. 6. The geometric interpretation of the spline evaluation algorithm.

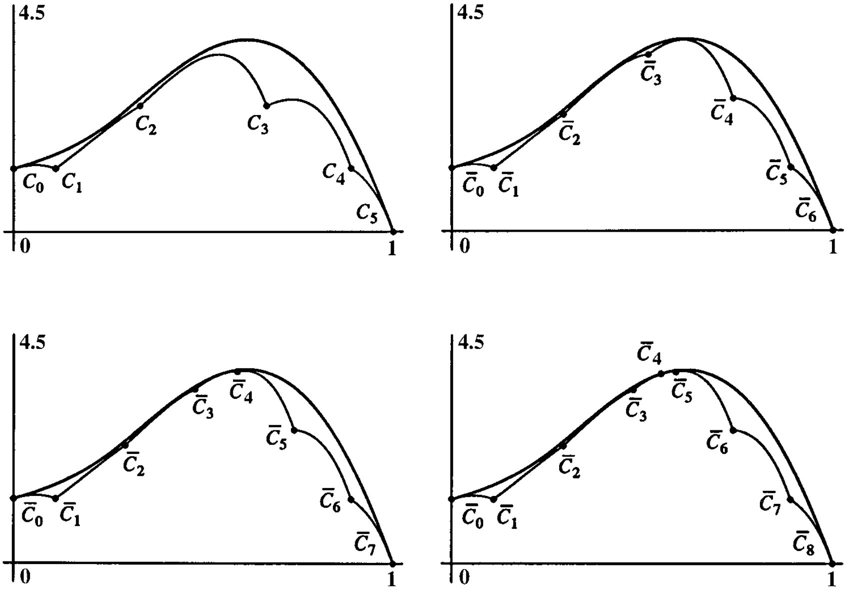

FIG. 7. Multiple knot insertion with full knot multiplicity.

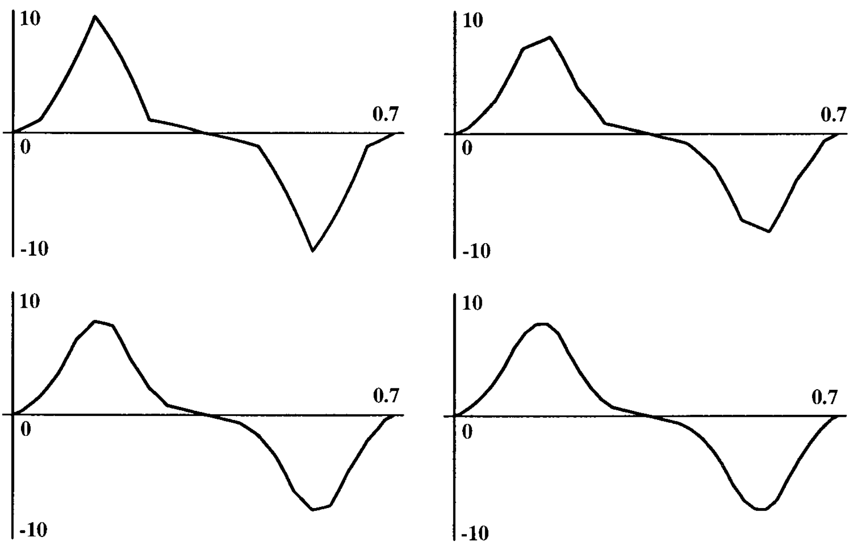

FIG. 8. Uniform subdivision of a quadratic spline.

$\overline{C}_6 = C_4^1$, and $\overline{C}_7 = C_4$. Therefore the evaluation algorithm can be viewed as a special case of knot insertion.

Finally, as Fig. 8 shows, as more knots are inserted into the spline, the refined control curves converge to the original smooth spline curve. The subdivided curve is a quadratic spline with knots $0, 0, 0, 1/10, \ldots, 6/10, 7/10, 7/10, 7/10$ and coefficients $0, 1, 9, 1, 0, -1, -9, -1, 0$. The subdivision is achieved by inserting the new knots half way between the old knots. The control curve corresponding to a refined knotvector clearly converges quite rapidly to the spline function.

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